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SIMPLICIAL GROUP MODELS FOR $\Omega^n S^n X$†

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ABSTRACT

Let X be a pointed simplicial set. The free group functors F [10] and Γ [1] provide simplicial models of $\Omega S|X|$ and $\Omega^\infty S^\infty|X|$. The simplicial group FX is a simplicial subgroup of ΓX, and this corresponds to the inclusion $\Omega S|X| \subset \Omega^\infty S^\infty X$. In this paper we define free group functors $\Gamma^{(n)}$ such that $\Gamma^{(n)}X$ is a model of $\Omega^n S^n|X|$. Moreover, there is natural filtration

$$FX = \Gamma^{(1)}X \subset \Gamma^{(2)}X \subset \cdots \subset \Gamma^{(n)}X \subset \cdots \subset \Gamma X,$$

corresponding to the filtration

$$\Omega S|X| \subset \Omega^2 S^2|X| \subset \cdots \subset \Omega^n S^n|X| \subset \cdots \subset \Omega^\infty S^\infty|X|.$$

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§1. Introduction

Let Y be a pointed space and let n be a positive integer. Let $\Omega^n S^n Y$ denote the n-fold loops on the n-fold (reduced) suspension of Y. There is a natural inclusion $\Omega^n S^n Y \subset \Omega^{n+1} S^{n+1} Y$ given by

$$\Omega^n i_{S^n Y} : \Omega^n S^n Y \to \Omega^n \Omega S S^n Y$$

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where $i_Y: Y \to \Omega SY$ is the adjoint of the identity map of the suspension $SY \to SY$. The union of the chain of inclusions

$$\Omega SY \subset \Omega^2 S^2 Y \subset \cdots \subset \Omega^n S^n Y \cdots$$

is denoted $\Omega^\infty S^\infty Y$ and is an infinite loop space. Several authors have given topological models of the loop spaces $\Omega^n S^n Y$ and $\Omega^\infty S^\infty Y$ (for example James [6], May [8], Milgram [9]). In this paper we give simplicial models of these loop spaces.

Simplicial models have been given for the two extreme cases, ΩSY and $\Omega^\infty S^\infty Y$. Milnor's functor F [10] takes a pointed simplicial set X to a free simplicial group FX such that the geometric realization $|FX|$ is naturally homotopy equivalent to $\Omega S|X|$. At the other extreme the functor Γ, defined by Barratt and Eccles [1], takes X to a simplicial group ΓX such that the realization $|\Gamma X|$ is naturally homotopy equivalent to $\Omega^\infty S^\infty |X|$. Moreover, FX is a simplicial subgroup of ΓX, and the inclusion $FX \subset \Gamma X$ corresponds naturally to the inclusion $\Omega S|X| \subset \Omega^\infty S^\infty |X|$.

In this paper we fill the gap between F and Γ by defining functors $\Gamma^{(n)}$ from pointed simplicial sets to free simplicial groups. We will obtain a natural filtration of ΓX by simplicial subgroups

$$\Gamma^{(1)}X \subset \Gamma^{(2)}X \subset \cdots \subset \Gamma^{(n)}X \subset \cdots \subset \Gamma X$$

such that the realization $|\Gamma^{(n)}X|$ is naturally homotopy equivalent to $\Omega^n S^n |X|$.

The main result is:

1.1. THEOREM. *Let X be a pointed simplicial set and let $n \geqq 1$ be an integer. There exists a natural "zigzag" of weak (homotopy) equivalences (defined in* 3.11*) connecting the simplicial groups* $\Gamma^{(n)}X$ and $G^n E^n X$ *(where E* (3.5) *is the simplicial suspension functor and G* (3.6) *is the functor that assigns to a reduced simplicial set K a free simplicial group GK having the homotopy type of the loops on K). Moreover, the inclusion $\Gamma^{(n)}X \subset \Gamma^{(n+1)}X$ corresponds naturally to the inclusion $G^n E^n X \subset G^{n+1} E^{n+1} X$ defined in* 3.11.

The exact relationship between the loop functors G and Ω, and between the suspension functors E and S is discussed in the appendix. This discussion and 1.1 immediately imply

1.2. COROLLARY. *Let X be a pointed simplicial set and let $n \geqq 1$ be an integer. The geometric realization $|\Gamma^{(n)}X|$ is naturally homotopy equivalent to*

$\Omega^n S^n |X|$. Moreover, the inclusion $|\Gamma^{(n)}X| \subset |\Gamma^{(n+1)}X|$ corresponds naturally to the inclusion $\Omega^n S^n |X| \subset \Omega^{n+1} S^{n+1} |X|$.

1.3. *Organization of the paper.* We work simplicially. The basic results of simplicial homotopy theory can be found in [7].

We define (in §2) functors $\Gamma^{(n)+}$ from pointed simplicial sets to free simplicial monoids. The functors $\Gamma^{(n)}$ (3.2) are the group completions (3.1) of the functors $\Gamma^{(n)+}$. The configuration complexes $C_n\Sigma_p$ (2.7) are the crucial ingredient in the definitions of $\Gamma^{(n)+}$ and of $\Gamma^{(n)}$. The main result is proved in §3 using Lemma 3.8 which is proved in §§4–5.

The author would like to thank his friend and advisor D. M. Kan for many helpful discussions.

§2. The functors $\Gamma^{(n)+}$

In this section we recall the definition of the functor Γ^+ [1, 3.1], and use the same type of construction to define, for each integer $n \geqq 1$, a functor $\Gamma^{(n)+}$. The properties of the functors Γ^+ (defined in 2.6) and $\Gamma^{(n)+}$ (defined in 2.9), which will arise in this section, are collected in the following proposition:

2.1. PROPOSITION. *Let X be a pointed simplicial set and let $n \geqq 1$ be an integer.*

(i) Γ^+ (2.6) *and* $\Gamma^{(n)+}$ (2.9) *are functors from pointed simplicial sets to free simplicial monoids.*

(ii) *$\Gamma^{(n)+}X$ is a simplicial submonoid of $\Gamma^{(n+1)+}X$, and the union of the chain of inclusions*

$$\Gamma^{(1)+}X \subset \Gamma^{(2)+}X \subset \cdots \subset \Gamma^{(n)+}X \subset \cdots$$

is the simplicial monoid Γ^+X.

(iii) *The basis of the free simplicial monoid $\Gamma^{(n)+}X$ is a subset of the basis of $\Gamma^{(n+1)}X$ and the union of these bases for all $n \geqq 1$ is the basis* (2.14) *of Γ^+X.*

2.2. *The functor W.* This is a functor from sets to simplicial sets. Let A be a set and let $f: A \to B$ be a map of sets. The simplicial set WA is defined by

$$(WA)_{k+1} = A^{k+1} = \{(a_0, a_1, \ldots, a_k) \mid a_i \in A\},$$

$$\partial_i(a_0, a_1, \ldots, a_k) = (a_0, \ldots, \hat{a}_i, \ldots, a_k) \qquad \text{(i.e. omit } a_i\text{)},$$

$$s_i(a_0, a_1, \ldots, a_k) = (a_0, \ldots, a_i, a_i, \ldots, a_k) \qquad \text{(i.e. repeat } a_i\text{)}.$$

The simplicial map Wf: $WA \to WB$ is defined by

$$Wf(a_0, a_1, \ldots, a_k) = (f(a_0), f(a_1), \ldots, f(a_k)).$$

2.3. *The permutation groups.* Let $J \subset \{1, 2, \ldots\}$ be a finite set of positive integers. Let Σ_J denote the group of permutations of J. In the special case that $J = \{1, 2, \ldots, p\}$, let Σ_p denote the permutation group Σ_J of the set $J = \{1, 2, \ldots, p\}$. In particular $\Sigma_0 = \Sigma_\phi$ is the trivial group.

Suppose that J contains j elements. An ordering of J is a sequence $(t_1, t_2, \ldots, t_j)$ that lists each integer in J exactly once. We identify the group Σ_J with the collection of orderings of J, identifying the permutation $\alpha \in \Sigma_J$ with the ordering $(t_1, t_2, \ldots, t_j)$ such that $\alpha(t_1) < \alpha(t_2) < \cdots < \alpha(t_j)$.

2.4. *The reduction maps.* Let J and K be finite sets such that $J \subset K \subset \{1, 2, \ldots\}$. *The reduction of an ordering* $(t_1, t_2, \ldots, t_k)$ of K to an ordering of J is obtained by taking the subsequence $(t_1, t_2, \ldots, t_j)$ consisting of the integers in J. Then the *reduction map of permutations* $R_J^K : \Sigma_K \to \Sigma_J$ is defined using the identification of permutations and orderings (2.3). Applying the functor W to the map $R_J^K : \Sigma_K \to \Sigma_J$, we define the *reduction map* R_J^K: $W\Sigma_K \to W\Sigma_J$. When it is clear which reduction map is meant, we abbreviate R_J^K to R_J, or to R.

2.5. *The diagonal action.* Let α_i, $\beta \in \Sigma_J$ be permutations. The diagonal action of Σ_J on $W\Sigma_J$ is defined by

$$(\alpha_0, \alpha_1, \ldots, \alpha_k) \cdot \beta = (\alpha_0 \circ \beta, \alpha_1 \circ \beta, \ldots, \alpha_k \circ \beta).$$

2.6. *The functor* Γ^+ [1, 3.1]. Let X be a pointed simplicial set and let $* \in X_0$ be the basepoint. Let VX denote the disjoint union $\bigcup_{p \geq 0} W\Sigma_p \times X^p$ where X^p is the p-fold cartesian product. The equivalence relation $\sim$, generated by the following relations, respects the face and degeneracy maps of VX; we define the *simplicial set* $\Gamma^+ X$ to be the quotient simplicial set $VX/\sim$. Let $w \in W\Sigma_p$ and $x_i \in X_k$ be k-simplices, let $\alpha \in \Sigma_p$ be a permutation, and let $R: W\Sigma_p \to W\Sigma_{p-1}$ be the reduction map (2.4). The relations generating $\sim$ are

(a) $(w, x_1, x_2, \ldots, x_p) \sim (w \cdot \alpha, x_{\alpha(1)}, x_{\alpha(2)}, \ldots, x_{\alpha(p)})$,

(b) $(w, x_1, \ldots, x_{p-1}, *) \sim (R(w), x_1, \ldots, x_{p-1})$.

The equivalence class of a simplex $(w, x_1, \ldots, x_p) \in W\Sigma_p \times X^p$ is denoted $[w, x_1, \ldots, x_p]$. The canonical choice for the basepoint of $\Gamma^+ X$ is the equivalence class $[1, \phi]$ of the 0-simplex $(1, \phi) \in W\Sigma_0 \times X^0$.

Let $f: X \to Y$ be a pointed simplicial map. The simplicial map $\Gamma^+ f$: $\Gamma^+ X \to \Gamma^+ Y$ is defined by

$$\Gamma^+ f[w, x_1, \dots, x_p] = [w, f(x_1), \dots, f(x_p)].$$

At this point we have defined Γ^+ as a functor from pointed simplicial sets to pointed simplicial sets.

2.7. *The configuration complexes.* Recall that the n-skeleton, $\mathrm{sk}_n X$, of a simplicial set X is the simplicial subset of X generated by the simplices of dimensions $\leqq n$. Let $p \geqq 0$ be an integer and let $\{i, j\}$ be a pair of integers such that $1 \leqq i < j \leqq p$. In 2.4 we defined a reduction map $R_{\{i,j\}} : W\Sigma_p \to W\Sigma_{\{i,j\}}$. The *configuration complex* $C_n\Sigma_p$ $(n \geqq 1)$ is the intersection of the pull backs

$$R_{\{i,j\}}^{-1}(\mathrm{sk}_{n-1} W\Sigma_{\{i,j\}})$$

for all pairs $\{i, j\} \subset \{1, 2, \dots, p\}$.

The reason for the name "configuration complex" is that it seems likely that the realization $|C_n\Sigma_p|$ is homotopically equivalent to $F(R^n, p)$, the configuration space of p distinct points in R^n.

The configuration complexes $C_n\Sigma_p$ are used (2.9) in the definition of the functor $\Gamma^{(n)+}$. The properties of $C_n\Sigma_p$ which we will need are

2.8. Proposition. *Let $p \geqq 0$ and $n \geqq 1$ be integers.*

(i) *The simplicial subset $C_n\Sigma_p \subset W\Sigma_p$ is closed under the action of Σ_p.*

(ii) *The simplicial subset $C_n\Sigma_p$ is contained in $C_{n+1}\Sigma_p$ and the union $\bigcup_{n \geqq 1} C_n\Sigma_p$ is $W\Sigma_p$.*

(iii) *The reduction map $R : W\Sigma_p \to W\Sigma_{p-1}$ restricts to a map of simplicial subsets $R : C_n\Sigma_p \to C_n\Sigma_{p-1}$.*

Proof. To prove (i) we note that the following two statements are equivalent:

(a) $R_{\{i,j\}}(w \cdot \alpha) \in \mathrm{sk}_{n-1} W\Sigma_{\{i,j\}}$,

(b) $R_{\{\alpha(i), \alpha(j)\}}(w) \in \mathrm{sk}_{n-1} W\Sigma_{\{\alpha(i), \alpha(j)\}}$,

where $w \in W\Sigma_p$ is a simplex and $\alpha \in \Sigma_p$ is a permutation.

Statements (ii) and (iii) are straightforward consequences of Definitions 2.4 and 2.7.

2.9. *The functor* $\Gamma^{(n)+}$. Let X be pointed simplicial set and let $* \in X_0$ denote the basepoint. Let $V^{(n)}X$ denote the disjoint union

$$V^{(n)}X = \bigcup_{p \geqq 0} C_n\Sigma_p \times X^p.$$

It follows from 2.8(i) and (iii) that $V^{(n)}X$ is a simplicial subset of VX and that $V^{(n)}X$ is closed under the equivalence relation $\sim$ (2.6). The simplicial set $\Gamma^{(n)+}X$ is the quotient simplicial set $V^{(n)}X/\sim$. Clearly $\Gamma^{(n)+}X$ is a simplicial subset of $\Gamma^+X = VX/\sim$, and it follows from 2.8(ii) that Γ^+X is filtered as follows:

$$\Gamma^{(1)+}X \subset \Gamma^{(2)+}X \subset \cdots \subset \Gamma^{(n)+}X \subset \cdots \subset \Gamma^+X.$$

Let $f: X \to Y$ be a pointed simplicial map. The simplicial map Γ^+f: $\Gamma^+X \to \Gamma^+Y$ (2.6) restricts to a simplicial map $\Gamma^{(n)+}X \to \Gamma^{(n)+}Y$ and thus $\Gamma^{(n)+}$ is a functor from pointed simplicial sets to pointed simplicial sets.

The definition (2.12) of the product on Γ^+X uses

2.10. *The product* $W\Sigma_p \times W\Sigma_q \to W\Sigma_{p+q}$. Let $p \geqq 0$ and $q \geqq 0$ be integers. The cartesian product $\Sigma_p \times \Sigma_q$ is identified with a subgroup of Σ_{p+q} as follows. For permutations $\alpha \in \Sigma_p$ and $\beta \in \Sigma_q$, the permutation $\alpha \times \beta \in \Sigma_{p+q}$ is defined by

$$\begin{aligned} \alpha \times \beta(i) &= \alpha(i), & i &\in \{1, 2, \ldots, p\}, \\ \alpha \times \beta(j+p) &= \beta(j) + p, & j &\in \{1, 2, \ldots, q\}. \end{aligned}$$

The product $W\Sigma_p \times W\Sigma_q \to W\Sigma_{p+q}$ is the simplicial map defined as follows. For k-simplices

$$w = (\alpha_0, \alpha_1, \ldots, \alpha_k) \in W\Sigma_p \quad \text{and} \quad v = (\beta_0, \beta_1, \ldots, \beta_k) \in W\Sigma_q,$$

let $w \times v \in W\Sigma_{p+q}$ be given by

$$w \times v = (\alpha_0 \times \beta_0, \alpha_1 \times \beta_1, \ldots, \alpha_k \times \beta_k).$$

2.11. PROPOSITION. *Let $w \in W\Sigma_p$, $v \in W\Sigma_q$ and $u \in W\Sigma_r$ be k-simplices, $k \geqq 0$.*

(i) *The product* (2.10) *is associative, i.e.* $(w \times v) \times u = w \times (v \times u) \in W\Sigma_{p+q+r}$.

(ii) *The product $w \times v \in W\Sigma_{p+q}$ is in the simplicial subset $C_n\Sigma_{p+q} \subset W\Sigma_{p+q}$ if and only if both $w \in C_n\Sigma_p$ and $v \in C_n\Sigma_q$.*

The proof is a straightforward computation using Definitions 2.7 and 2.10.

2.12. *The product on Γ^+X* [1, 3.9]. Let $[w, x_1, \ldots, x_p]$ and $[v, y_1, \ldots, y_q]$ be k-simplices ($k \geqq 0$) of Γ^+X. Their product is given by

$$[w \times v, x_1, \ldots, x_p, y_1, \ldots, y_q].$$

2.13. Corollary of 2.11. *Let X be a pointed ($*\in X_0$) simplicial set.*

(i) *The simplicial set Γ^+X with the product defined in 2.12 is a simplicial monoid; its unit element in dimension k is $s_0^k[1,\phi]$, where $[1,\phi]$ is the basepoint of Γ^+X (2.6).*

(ii) *For k-simplices $a,b\in\Sigma^+X$, the product $a\cdot b\in\Gamma^+X$ is in the simplicial subset $\Gamma^{(n)+}X$ if and only if both $a\in\Gamma^{(n)+}X$ and $b\in\Gamma^{(n)+}X$. In particular $\Gamma^{(n)+}X$ is a simplicial submonoid of Γ^+X.*

The map $\Gamma^+f:\Gamma^+X\to\Gamma^+Y$ (2.6) and its restriction $\Gamma^{(n)+}f:\Gamma^{(n)+}X\to\Gamma^{(n)+}Y$ (2.9) are simplicial homomorphisms, and therefore Γ^+ and $\Gamma^{(n)+}$ are functors to simplicial monoids. The next proposition and corollary show that Γ^+ and $\Gamma^{(n)+}$ are in fact functors to free simplicial monoids.

2.14. Proposition [1, 3.11]. *Let X be a pointed ($*\in X_0$) simplicial set. The simplicial monoid Γ^+X is a free simplicial monoid.*

Proof. Using relations (a) and (b) of 2.6, it follows that each k-simplex of Γ^+X can be put in the form $[w,x_1,\dots,x_p]$ with $w\in W\Sigma_p$ and $x_i\in X_k$ k-simplices such that $x_i\neq s_0^k*$, $1\leqq i\leqq p$. The simplex $[w,x_1,\dots,x_p]$ is said to be irreducible if there is *no* permutation $\alpha\in\Sigma_p$ such that $w\cdot\alpha\in W\Sigma_p$ is in the product $W\Sigma_r\times W\Sigma_{p-r}\subset W\Sigma_p$ for some integer r, $1\leqq r\leqq p-1$. It is proved in [1, 3.11] that each k-simplex can be written uniquely as a product of irreducible k-simplices. Notice that a degeneracy of an irreducible simplex is also irreducible. Therefore Γ^+X is a free simplicial monoid and its basis is the set of irreducible simplices of Γ^+X.

Propositions 2.14 and 2.13(ii) imply

2.15. Corollary. *The simplicial submonoid $\Gamma^{(n)+}X\subset\Gamma^+X$ is a free simplicial monoid and its basis is the set of simplices $\Gamma^{(n)+}X$ which are irreducible in Γ^+X.*

2.16. Remark. It follows from 2.7 that $C_1\Sigma_p$ is the 0-skeleton of $W\Sigma_p$ which is the discrete simplicial set Σ_p (i.e. it has Σ_p in each dimension and the face and degeneracy maps are the identity map $\Sigma_p\to\Sigma_p$). The irreducible k-simplices of $\Gamma^{(1)+}X$ (and thus its basis) are the k-simplex of X and $1\in\Sigma_1$ is the identity permutation of $\{1\}$. Therefore the simplicial monoid $\Gamma^{(1)+}X$ coincides with Milnor's [10] simplicial monoid F^+X which is the simplicial version of the James construction.

§3. The functors $\Gamma^{(n)}$ and the main result

In this section we define, for each integer $n \geqq 1$, a functor $\Gamma^{(n)}$ from pointed simplicial sets to free simplicial groups, and we prove the main result (1.1). The proof uses Lemma 3.8 which will be proved in §§4–5.

3.1. *The group completion.* Let M be a monoid and let e denote its unit. The group completion of M is defined by adjoining inverses to M as follows. Let FM be the free group with a generator $\bar{m}$ for each element $m \in M$ and with $\bar{e}$ identified with the unit of FM. Let N be the normal subgroup of FM that is generated by the elements of the form $\bar{u}\bar{v}\overline{uv}^{-1}$ where $u, v \in M$. The *group completion* of M is the quotient group $UM = FM/N$.

The *group completion of a homomorphism* of monoids $h: M \to M'$ is the homomorphism of groups $Uh: UM \to UM'$ induced by the natural homomorphism of free groups $Fh: FM \to FM'$.

The *group completion of a simplicial* monoid M is the natural simplicial group UM defined by taking the group completions of the monoids M_k and of the homomorphisms $\partial_i: M_k \to M_{k-1}$ and $S_i: M_k \to M_{k+1}$.

3.2. *The functors* Γ *and* $\Gamma^{(n)}$. The *functor* Γ [1, 4.3] is the composition of functors $U\Gamma^+$, and the *functor* $\Gamma^{(n)}$ is the composition $U\Gamma^{(n)+}$.

The definition and Proposition 2.1 immediately imply

3.3. Proposition. *Let X be a pointed simplicial set and let $n \geqq 1$ be an integer.*

(i) *Γ and $\Gamma^{(n)}$ are functors from pointed simplicial sets to free simplicial groups. A basis of ΓX ($\Gamma^{(n)}X$) is the set of irreducible simplices of ΓX ($\Gamma^{(n)+}X$).*

(ii) *$\Gamma^{(n)}X$ is a simplicial subgroup of $\Gamma^{(n+1)}X$ and the union of the chain of inclusions*

$$\Gamma^{(1)}X \subset \Gamma^{(2)}X \subset \cdots \subset \Gamma^{(n)}X \subset \cdots$$

is the simplicial group ΓX.

3.4. Remark. It follows from Remark 2.16 that the functor $\Gamma^{(1)}$ coincides with Milnor's functor F [10]. A natural inclusion $\iota_X: X \to \Gamma^{(1)}X$ is defined on the k-simplices $x \in X_k$ by the formula

$$\iota_X(x) = [s_0^k 1, x].$$

3.5. *The functors C and E.* Let X be a pointed ($* \in X_0$) simplicial set. The *cone* on X is the simplicial set CX defined as follows. The k-simplices of CX are the pairs (p, x), where p is an integer $1 \leqq p \leqq k$, and $x \in X_{k-p}$ is a simplex, and where $(p, s_0^{k-p}*)$ is identified with $(k, *)$. The face and degeneracy operators on CX are defined by the formulas

$$\begin{aligned}
s_i(p,x) &= (p+1,x), && i<p,\\
s_i(p,x) &= (p, s_{i-p}x), && i \geqq p,\\
\partial_i(p,x) &= (p-1,x), && i<p,\\
\partial_i(p,x) &= (p, \partial_{i-p}x), && i \geqq p,\\
\partial_1(1,x) &= (0,*), && x \in X_0.
\end{aligned}$$

To justify calling CX the cone on X we observe that:

(i) There is a natural *inclusion of* X *into* CX, identifying the simplex $x \in X$ with the simplex $(0,x) \in CX$.

(ii) The simplicial set CX is *contractible*.

(A contracting homotopy is given in 5.4.)

Let $f: X \to Y$ be a pointed simplicial map. A natural simplicial map $Cf: CX \to CY$ is defined by

$$Cf(p,x) = (p, f(x)).$$

The *suspension of* X is the natural quotient simplicial set $EX = CX/X$.

3.6. *The functor* G [7, p. 118]. Let X be a reduced simplicial set, i.e. X has a unique 0-simplex $* \in X_0$. The simplicial group GX is a loop group of X and is defined as follows. The group of k-simplices G_kX is the group that has

(i) one generator $\bar{x}$ for each $(k+1)$-simplex $x \in X_{k+1}$,

(ii) one relation $\overline{s_0y} = e_k$ (the unit of G_kX) for each k-simplex $y \in X_k$.

Clearly the groups G_kX are free. Thus the face and degeneracy homomorphisms $\partial_i: G_kX \to G_{k-1}X$ and $s_i: G_kX \to G_{k+1}X$ are defined by the following formulas on the generators

$$\begin{aligned}
\partial_0\bar{x} &= \overline{\partial_0 x}^{-1} \cdot \overline{\partial_1 x} && \\
\partial_i\bar{x} &= \overline{\partial_{i+1} x}, && 1 \leqq i \leqq k,\\
s_i\bar{x} &= \overline{s_{i+1} x}, && 0 \leqq i \leqq k.
\end{aligned}$$

Let $f: X \to Y$ be a map of reduced simplicial sets. A natural simplicial homomorphism $Gf: GX \to GY$ is defined on the generators by the formula

$$Gf(\bar{x}) = \overline{f(x)}.$$

The first step in proving 1.1 is

3.7. Lemma. *Let X be a pointed simplicial set and let $n \geqq 1$ be an integer. There exists a simplicial group $K^{(n)}X$* (defined in §4) *and natural simplicial homomorphisms*

$$\Gamma^{(n+1)}X \to K^{(n)}X \leftarrow G\Gamma^{(n)}EX$$

which are weak equivalences.

The proofs of 3.7 and 1.1 use Lemma 3.8 which will be proved in §§4–5.

3.8. Lemma. *Let X be a pointed simplicial set and let $g: CX \to EX$ be the natural quotient map* (3.5). *There is a natural filtration* (*defined in* §4) *of ΓCX by simplicial subgroups*

$$T^{(1)}X \subset T^{(2)}X \subset \cdots \subset T^{(n)}X \subset \cdots \subset \Gamma CX$$

such that:

(i) *$\Gamma^{(n)}CX$ and $\Gamma^{(n+1)}X$ are simplicial subgroups of $T^{(n)}X$ and $T^{(n)}X$ is a simplicial subgroup of $\Gamma^{(n+1)}CX$.*

(ii) *The simplicial homomorphism $\Gamma g: \Gamma CX \to \Gamma EX$ restricts to a homomorphism $\gamma^{(n)}: T^{(n)}X \to \Gamma^{(n)}EX$.*

It follows from (i) and (ii) that $\Gamma^{(n+1)}X$ is a simplicial subgroup of the kernel of the homomorphism $\gamma^{(n)}: T^{(n)}X \to \Gamma^{(n)}EX$, which is denoted $K^{(n)}X = \ker \gamma^{(n)}$.

(iii) *The inclusion $\Gamma^{(n+1)}X \subset K^{(n)}X$ is a weak equivalence.*

(iv) *$T^{(n)}X$ is contractible* (*i.e. it is connected and its homotopy groups vanish*).

3.9. Proof of 3.7. The simplicial group $K^{(n)}X$ is the kernel of $\gamma^{(n)}: T^{(n)}X \to \Gamma^{(n)}EX$. The natural simplicial homomorphism $\Gamma^{(n+1)}X \to K^{(n)}X$ of 3.7 is the inclusion $\Gamma^{(n+1)}X \subset K^{(n)}X$ which, by 3.8(iii), is a weak equivalence.

To define the natural simplicial homomorphism $G\Gamma^{(n)}EX \to K^{(n)}X$ consider the principal fibration

$$K^{(n)}X \to T^{(n)}X \to \Gamma^{(n)}EX.$$

The total space $T^{(n)}X$ is contractible and it follows from [7, p. 123] that there exists a simplicial homomorphism (not uniquely)

$$h^{(n)}: G\Gamma^{(n)}EX \to K^{(n)}X$$

which is a weak equivalence. The procedure for defining such an $h^{(n)}$ is as follows. Find a pseudo-cross section [7, p. 73] (i.e. a cross section commuting with all faces and degeneracies other than ∂_0) $\sigma^{(n)}: \Gamma^{(n)}EX \to T^{(n)}X$ of the homomorphism $\gamma^{(n)}: T^{(n)}X \to \Gamma^{(n)}EX$. Then define $h^{(n)}$ on the generators of the free simplicial group $G\Gamma^{(n)}EX$ by the formula

$$h^{(n)}(\bar{a}) = \sigma^{(n)}(\partial_0 a)^{-1} \cdot \sigma^{(n)}(\partial_1 a),$$

where $a \in \Gamma^{(n)}EX$ is a simplex.

Therefore to define a *natural simplicial homomorphism* $h^{(n)}: G\Gamma^{(n)}EX \to K^{(n)}X$ it suffices to define a *natural pseudo-cross section* $\sigma^{(n)}: \Gamma^{(n)}EX \to T^{(n)}X$. Note that there is a pseudo-cross section $r: EX \to CX$ of the quotient map $g: CX \to EX$ given by

$$rg(p, x) = (p, x), \qquad p \geqq 1$$

where $(p, x) \in CX$ is a simplex. The pseudo-cross section $\sigma^{(n)}: \Gamma^{(n)}EX \to T^{(n)}X$ is the composition $\Gamma^{(n)}EX \to \Gamma^{(n)}CX \subset T^{(n)}X$ of $\Gamma^{(n)}r: \Gamma^{(n)}EX \to \Gamma^{(n)}CX$ (which makes sense even though r is not a simplicial map) and of the inclusion $\Gamma^{(n)}CX \subset T^{(n)}X$ (3.8).

3.10. Lemma. *Let X be a pointed simplicial set. The simplicial groups $\Gamma^{(1)}X$ and GEX are naturally isomorphic.*

Proof. Let $(1, x) \in CX$ be a simplex and let $y \in EX$ denote the image $g(1, x)$, where $g: CX \to EX$ is the quotient map. The isomorphism $f: \Gamma^{(1)}X \to GEX$ is defined on the generators of $\Gamma^{(1)}X$ by the formula

$$f[s_0^{k_*}, x] = \bar{y}.$$

We now complete the

3.11. Proof of 1.1. The infinite diagram defined below proves 1.1. It is a natural commuting diagram of simplicial homomorphisms and simplicial groups. The nth row is the "zigzag" of weak equivalences connecting the simplicial groups $\Gamma^{(n)}X$ and G^nE^nX. The slanting arrows on the right hand side of the diagram are the inclusions $G^nE^nX \subset G^{n+1}E^{n+1}X$ defined by

$$G^n\iota_{E^nX}: G^nE^nX \to G^nGEE^nX = G^{n+1}E^{n+1}X$$

where the inclusion $\iota_X: X \to GEX$ is obtained by combining 3.4 and 3.10. It follows from the commutativity of the diagram that the inclusion $\Gamma^{(n)}X \subset \Gamma^{(n+1)}X$ corresponds naturally to the inclusion $G^nE^nX \subset G^{n+1}E^{n+1}X$.

The diagram is

$$
\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\Gamma^{(4)}X \arrow[r] \arrow[u] & K^{(3)}X \arrow[u] & G\Gamma^{(3)}EX \arrow[l] \arrow[r] \arrow[u] & GK^{(2)}EX \arrow[u] & G^2\Gamma^{(2)}E^2X \arrow[l] \arrow[r] \arrow[u] & {} \\
\Gamma^{(3)}X \arrow[r] \arrow[u] & K^{(2)}X \arrow[u] & G\Gamma^{(2)}EX \arrow[l] \arrow[r] \arrow[u] & GK^{(1)}EX \arrow[u] & G^2\Gamma^{(1)}E^2X \arrow[l] \arrow[u] \arrow[d, equal] & \\
\Gamma^{(2)}X \arrow[r] \arrow[u] & K^{(1)}X \arrow[u] & G\Gamma^{(1)}EX \arrow[l] \arrow[u] & & G^3E^3X & \\
\Gamma^{(1)}X \arrow[u] & & G^2E^2X \arrow[u, "\mid"'] \arrow[urr] & & & \\
GEX \arrow[u, leftrightarrow, "\Vert"'] \arrow[urr] & & & & &
\end{tikzcd}
$$

(a) (b) (c) (d) (e)

The construction of this diagram requires comment. The basic idea is that columns (a), (b) and (c) are constructed using 3.7 and 3.8, and that (a), (b) and (c) generate the rest of the diagram by "replication".

The horizontal maps from column (a) to column (b) and the maps from (c) to (b) come from Lemma 3.7 and are, therefore, weak equivalences. The vertical maps in (a) and (b) are the inclusions given in 3.4 and 3.8. The vertical maps in (c) are defined by applying the functor G to the natural inclusions $\Gamma^{(n)}EX \subset \Gamma^{(n+1)}EX$. It follows from the definitions of the maps that the subdiagram generated by columns (a), (b) and (c) commutes.

To "replicate" this subdiagram on (a), (b) and (c) we replace X by its suspension EX and then apply the functor G to the diagram. The resulting diagram is the one generated by columns (c), (d) and (e). The procedure is repeated indefinitely to obtain the rest of the diagram. Since G preserves weak equivalence, it follows that all the horizontal maps of the diagram are weak equivalences.

§4. The relative functors

In this section we define relative functors $\Gamma^{(n)}$ ($\Gamma^{(n)+}$) from pointed pairs (X, A) of simplicial sets to free simplicial groups (monoids). The simplicial group $T^{(n)}X$ (3.8) is defined to be $\Gamma^{(n)}(CX, X)$ where CX is the cone on X (3.5). The properties of $T^{(n)}X = \Gamma^{(n)}(CX, X)$ given in Lemma 3.8(i), (ii) and (iii) are special cases of properties (4.5 and 4.7) of the free simplicial groups $\Gamma^{(n)}(X, A)$. Lemma 3.8(iv) will be proved in §5.

4.1. *The simplicial subsets* $\mathrm{sk}_n^+ W\Sigma_{\{i,j\}}$ *and* $\mathrm{sk}_n^- W\Sigma_{\{i,j\}}$. Let $n \geqq 1$ be an integer and let $\{i,j\}$ be a pair of integers. The group $\Sigma_{\{i,j\}}=\{1,\alpha\}$ is a two element group, and so the simplicial set $\mathrm{sk}_n W\Sigma_{\{i,j\}}$ has exactly two non-degenerate n-simplices, $(1,\alpha,1,\ldots)$ and $(\alpha,1,\alpha,\ldots)$. The simplicial subset $\mathrm{sk}_n^+ W\Sigma_{\{i,j\}}\subset \mathrm{sk}_n W\Sigma_{\{i,j\}}$ is the one that is generated by the non-degenerate n-simplex $(1,\alpha,1,\ldots)$. The simplicial subset $\mathrm{sk}_n^- W\Sigma_{\{i,j\}}\subset \mathrm{sk}_n W\Sigma_{\{i,j\}}$ is generated by the non-degenerate n-simplex $(\alpha,1,\alpha,\ldots)$.

An immediate consequence of 4.1 is

4.2. Proposition. *Given integers $n\geqq 1$, i, and j, then*

(i) $\mathrm{sk}_n^+ W\Sigma_{\{i,j\}}\cap \mathrm{sk}_n^- W\Sigma_{\{i,j\}}=\mathrm{sk}_{n-1} W\Sigma_{\{i,j\}}$,

(ii) $\mathrm{sk}_n^+ W\Sigma_{\{i,j\}}\cup \mathrm{sk}_n^- W\Sigma_{\{i,j\}}=\mathrm{sk}_n W\Sigma_{\{i,j\}}$.

4.3. *The free simplicial monoid $\Gamma^{(n)+}(X,A)$.* Let (X,A) be a pointed pair of simplicial sets and let $n\geqq 1$ be an integer. The simplicial subset $\Gamma^{(n)+}(X,A)\subset \Gamma^{(n+1)+}X$ consists of those simplices $[w,x_1,x_2,\ldots,x_p]\in\Gamma^{(n+1)+}X$ such that, for integers i and j with $1\leqq i<j\leqq p$,

(a) $R_{\{i,j\}}(w)\in \mathrm{sk}_n^- W\Sigma_{\{i,j\}}$, if $x_i\in X-A$,

(b) $R_{\{i,j\}}(w)\in \mathrm{sk}_n^+ W\Sigma_{\{i,j\}}$, if $x_j\in X-A$.

As in 2.13(ii), the product $u\cdot v\in\Gamma^+X$ of k-simplices $u,v\in\Gamma^+X$ is in the simplicial subset $\Gamma^{(n)+}(X,A)$ if and only if both $u\in\Gamma^{(n)+}(X,A)$ and $v\in\Gamma^{(n)+}(X,A)$. Therefore, $\Gamma^{(n)+}(X,A)$ is a simplicial submonoid of Γ^+X, and $\Gamma^{(n)+}(X,A)$ is a free simplicial monoid, its basis is the set of irreducible simplices of Γ^+X (2.14) contained in $\Gamma^{(n)+}(X,A)$.

The definition of $\Gamma^{(n)+}(X,A)$ is natural for maps $f\colon(X,A)\to(Y,B)$ of pointed pairs of simplicial sets. The homomorphism $\Gamma^+f\colon\Gamma^+X\to\Gamma^+Y$ (2.6) restricts to a homomorphism $\Gamma^{(n)+}f\colon\Gamma^{(n)+}(X,A)\to\Gamma^{(n)+}(Y,B)$.

4.4. *The free simplicial group $\Gamma^{(n)}(X,A)$* is the group completion (3.1) of the free simplicial monoid $\Gamma^{(n)+}(X,A)$. It follows from the comments in 4.3 that the relative $\Gamma^{(n)}$ is a functor from pointed pairs of simplicial sets to free simplicial groups.

Definitions 4.4 and 4.3 and Proposition 4.2 together imply

4.5. Proposition. *Let (X,A) be a pointed pair and let $*\in A_0$ denote the basepoint. The following equalities and inclusions of simplicial groups hold:*

(i) $\Gamma^{(n)}(X,*)=\Gamma^{(n)}X$,

(ii) $\Gamma^{(n)}(X,X)=\Gamma^{(n+1)}X$,

(iii) $\Gamma^{(n)}X\subset\Gamma^{(n)}(X,A)\subset\Gamma^{(n+1)}X$,

(iv) $\Gamma^{(n+1)}A\subset\Gamma^{(n)}(X,A)$.

4.6. REMARK. Note that statements (i) and (ii) of Lemma 3.8 are corollaries of 4.5.

4.7. LEMMA. *Let (X, A) be a pointed pair of simplicial sets and let $f: (X, A) \to (X/A, *)$ denote the quotient map. Let* $\mathrm{Ker}^{(n)}(X, A)$ *denote the kernel of the homomorphism*

$$\Gamma^{(n)} f: \Gamma^{(n)}(X, A) \to \Gamma^{(n)}(X/A, *) = \Gamma^{(n)} X/A.$$

The inclusion $\Gamma^{(n+1)} A \subset \mathrm{Ker}^{(n)}(X, A)$ (which exists by 4.5(iv)) *is a weak equivalence.*

The homomorphism $\Gamma^{(n)}: T^{(n)} X \to \Gamma^{(n)} EX$ of 3.8 is $\Gamma^{(n)} g: \Gamma^{(n)}(CX, X) \to \Gamma^{(n)} EX$ where $g: (CX, X) \to (EX, *)$ is the quotient map (3.5).

4.8. COROLLARY. *The inclusion* $\Gamma^{(n+1)} X \to K^{(n)} X = \mathrm{Ker}^{(n)}(CX, X)$ *is a weak equivalence.*

The rest of this section is devoted to the proof of 4.7. The first part of the proof uses bisimplicial sets and a summary of their properties can be found in [2, §1]. In particular we will use the

4.9. HOMOTOPY INVARIANCE OF THE DIAGONAL [4, Ch. XII, §4]. *If $H \to L$ is a map of bisimplicial sets such that, for each integer $k \geqq 0$, the map at level p $H_{*,p} \to L_{*,p}$ is a weak equivalence, then the diagonal map* $\mathrm{Diag}\, H \to \mathrm{Diag}\, L$ *is also a weak equivalence.*

The first part of the proof is a reduction to the case that (X, A) is a pointed pair of discrete simplicial sets. This is done by realizing the inclusion $\Gamma^{(n+1)} A \subset \mathrm{Ker}^{(n)}(X, A)$ as the diagonal of a bisimplicial map and then using 4.9.

Note that $\mathrm{Ker}^{(n)}(X, A)$ is a functor from pointed pairs to simplicial groups. Let $L^{(n)}(X, A)$ denote the bisimplicial group that at level p is defined by

$$L^{(n)}(X, A)_{*,p} = \mathrm{Ker}^{(n)}(X_p, A_p).$$

The face and degeneracy maps between levels are the simplicial homomorphisms $\mathrm{Ker}^{(n)}(\partial_i)$ and $\mathrm{Ker}^{(n)}(s_i)$ where ∂_i and s_i are the face and degeneracy maps of (X, A).

Using the same construction, let $H^{(n+1)} A$ denote the bisimplicial group that at level p is defined by

$$(H^{(n+1)} A)_{*,p} = \Gamma^{(n+1)} A_p.$$

Let $H^{(n+1)}A \to L^{(n)}(X, A)$ denote the bisimplicial homomorphism that is the inclusion $\Gamma^{(n+1)}A_p \subset \mathrm{Ker}^{(n)}(X_p, A_p)$ at level p.

It follows that

Diag $L^{(n)}(X, A) = \mathrm{Ker}^{(n)}(X, A)$,

Diag $H^{(n+1)}A = \Gamma^{(n+1)}A$,

Diag$[H^{(n+1)}A \to L^{(n)}(X, A)]$ is the inclusion $\Gamma^{(n+1)}(X, A) \subset \mathrm{Ker}^{(n)}(X, A)$.

It follows from 4.9 that if the inclusion $\Gamma^{(n+1)}A_p \subset \mathrm{Ker}^{(n)}(X_p, A_p)$ at each level $p \geqq 0$ is a weak equivalence, then the inclusion $\Gamma^{(n+1)}A \subset \mathrm{Ker}^{(n)}(X, A)$ is also a weak equivalence.

So we can assume that (X, A) is a pointed pair of discrete simplicial sets. Therefore $X = A \vee B$ where B is the quotient X/A. Recall that $f: (A \vee B, A) \to (B, *)$ is the quotient map. Let $g: (A \vee B, A) \to (A, A)$ denote the map that smashes B to the basepoint.

The homomorphism $\Gamma^{(n)}g: \Gamma^{(n)}(A \vee B, A) \to \Gamma^{(n)}(A, A) = \Gamma^{(n+1)}A$ is a retraction onto the simplicial subgroup $\Gamma^{(n+1)}A \subset \Gamma^{(n)}(A \vee B, A)$. Hence the inclusion $\Gamma^{(n+1)}A \subset \mathrm{Ker}^{(n)}(A \vee B, A)$ is a weak equivalence if and only if the restriction $\Gamma^{(n)}g: \mathrm{Ker}^{(n)}(A \vee B, A) \to \Gamma^{(n+1)}A$ is a weak equivalence.

Consider the commuting diagram of fibrations:

$$\begin{array}{ccc} \mathrm{Ker}^{(n)}(A \vee B, A) & \Gamma^{(n)}(A \vee B, A) & \Gamma^{(n)}(B) \\ \Gamma^{(n)}g & \Gamma^{(n)}g \times \Gamma^{(n)}f & \\ \Gamma^{(n+1)}A & \Gamma^{(n+1)}A \times \Gamma^{(n)}B & \Gamma^{(n)}B \end{array}$$

The map $\Gamma^{(n)}g: \mathrm{Ker}^{(n)}(A \vee B, A) \to \Gamma^{(n+1)}A$ is a weak equivalence if and only if the homomorphism

$$\Gamma^{(n)}g \times \Gamma^{(n)}f: \Gamma^{(n)}(A \vee B, A) \to \Gamma^{(n+1)}A \times \Gamma^{(n)}B$$

is a weak equivalence.

4.10. Lemma. *The homomorphism $\Gamma^{(n)}g \times \Gamma^{(n)}f$ is a weak equivalence.*

Proof. Let M and M' be simplicial monoids and suppose that $h: M \to M'$ is a simplicial homomorphism which is a weak equivalence. In general it does not follow that the group completion $Uh: UM \to UM'$ is a weak equivalence. But if the monoids M and M' are "nice" (explained below) then it will follow that Uh is a weak equivalence. In particular this argument will apply to the homomorphism

$$\Gamma^{(n)+}g \times \Gamma^{(n)+}f: \Gamma^{(n)+}(A \vee B, A) \to \Gamma^{(n+1)+}A \times \Gamma^{(n)+}B,$$

the group completion of which is $\Gamma^{(n)}g \times \Gamma^{(n)}f$. Hence 4.10 will follow from 4.11.

Let $u: M \to UM$ the homomorphism that (dimensionwise) is the composition $M \to FM \to FM/N = UM$ of the quotient map and of the inclusion $M \to FM$ as generators. The simplicial monoid M is "nice" if the map of classifying complexes [7, p. 83] $\bar{W}u: \bar{W}M \to \bar{W}UM$ is a weak equivalence.

Consider the commuting diagram:

$$\begin{array}{ccccccc} UM & \to & WUM & \to & \bar{W}UM & \to & \bar{W}M \\ \downarrow{\scriptstyle Uh} & & \downarrow{\scriptstyle Wuh} & & \downarrow{\scriptstyle \bar{W}Uh} & & \downarrow{\scriptstyle \bar{W}h} \\ UM' & \to & WUM' & \to & \bar{W}UM' & \to & \bar{W}M' \end{array}$$

where $UM \to WUM \to \bar{W}UM$ is the natural principal fibration [7, p. 83] such that WUM is contractible. Since h is a weak equivalence it follows that $\bar{W}h$ is a weak equivalence. It the monoids M and M' are "nice", then it also follows that $\bar{W}Uh$ is a weak equivalence, and therefore that Uh is a weak equivalence.

It is known that free simplicial monoids are "nice" (see [5, 5.4]). Since the functors $\bar{W}$ and U commute with cartesian products, it follows that a cartesian product of free simplicial monoids is "nice". The simplicial monoids $\Gamma^{(n)+}(A \vee B, A)$, $\Gamma^{(n+1)}A$ and $\Gamma^{(n)+}B$ are free and therefore 4.10 follows from

4.11. Lemma. *The homomorphism* $\Gamma^{(n)+}g \times \Gamma^{(n)+}f$ *is a weak equivalence.*

The rest of this section is devoted to proving 4.11 which will finish the proof of 4.7.

4.12. *The filtration of* $C_{n+1}\Sigma_p$. Let $n \geqq 1$, p, and k be integers such that $0 \leqq k \leqq p$. The simplicial subset $E_k C_{n+1}\Sigma_p$ consists of the simplices $w \in C_{n+1}\Sigma_p$ such that

(a) $R_{\{i,j\}}(w) \in \mathrm{sk}_{n-1} W\Sigma_{\{i,j\}}$ when $k < i < j \leqq p$,

(b) $R_{\{i,j\}}(w) \in \mathrm{sk}_n^+ W\Sigma_{\{i,j\}}$ when $1 \leqq i \leqq k < j \leqq p$.

The following properties of the simplicial sets $E_k C_{n+1}\Sigma_p$ are immediate consequences of the definition.

4.13. Proposition. *Let* $p \geqq 0$ *and* $q \geqq 0$ *be integers.*

(i) *For* $0 \leqq k < p$, *the simplicial set* $E_k C_{n+1}\Sigma_p$ *is contained in* $E_{k+1}C_{n+1}\Sigma_p$. *At the extremes* $E_0 C_{n+1}\Sigma_p = C_n\Sigma_p$ *and* $E_p C_{n+1}\Sigma_p$ *is* $C_{n+1}\Sigma_p$.

(ii) *The simplicial subset* $E_pC_{n+1}\Sigma_{p+q} \subset C_{n+1}\Sigma_{p+q}$ *is closed under the diagonal action of the subgroup* $\Sigma_p \times \Sigma_q \subset \Sigma_{p+q}$.

(iii) *Using* 2.10, *the cartesian product* $C_{n+1}\Sigma_p \times C_n\Sigma_q$ *is identified with a* $\Sigma_p \times \Sigma_q$*-equivariant simplicial subset of* $E_pC_{n+1}\Sigma_{p+q}$.

4.14. REMARK. The simplicial subset $\Gamma^{(n)+}(X, A)$ of Γ^+X consists of the simplices in Γ^+X of the form $[w, a_1, \dots, a_p, x_1, \dots, x_q]$ where $w \in E_pC_{n+1}\Sigma_{p+q}$, $a_i \in A$ and $x_i \in X$ are simplices.

We now recall some facts about simplicial homotopies which are needed for the proof of 4.11.

4.15. *Simplicial homotopies*. The interval I is the simplicial set that has two 0-simplices 0 and 1, and a 1-simplex e such that $\partial_0 e = 0$ and $d_1 e = 1$. These simplices generate I. Thus

$$I_0 = \{0, 1\},$$

$$I_1 = \{0, 1, e\},$$

$$\dots$$

$$I_k = \{0, 1, \dots, s_{k-1} \dots \hat{s}_i \dots, s_0 e, \dots\}.$$

The realization $|I|$ is homeomorphic to the unit interval [0, 1].

A (simplicial) homotopy is a simplicial map $H: X \times I \to Y$. Let $H_0: X \to Y$ denote the composition $X = X \times \{0\} \subset X \times I \to Y$ of the homotopy and of the inclusion. Likewise $H_1: H \to Y$ denotes the composition $X \times \{1\} \to X \times I \to Y$.

PROOF OF 4.11. Since A and B are discrete, it follows that the simplicial monoid $\Gamma^{(n)+}(A \vee B, A)$ is a disjoint union $\bigcup_{p,q \ge 0} Z_{p,q}$ where $Z_{p,q}$ is defined by

$$Z_{p,q} = E_pC_{n+1}\Sigma_{p+q} \times_{\Sigma_p \times \Sigma_q} (A - *)^p \times (B - *)^q.$$

(Given a group G acting on simplicial set X and Y, let $X \times_G Y$ denote the orbit space of the diagonal action of G on $X \times Y$.)

Likewise the simplicial monoid $\Gamma^{(n+1)+}A$ is a disjoint union $\bigcup_{p,q \ge 0} C_{p,q}$ where $C_{p,q}$ is defined by

$$C_{p,q} = (C_{n+1}\Sigma_p \times C_n\Sigma_q) \times_{\Sigma_p \times \Sigma_q} (A - *)^p \times (B - *)^q.$$

Note that, by 4.3(iii), $C_{p,q}$ is a simplicial subset of $z_{p,q}$.

The homomorphism $\Gamma^{(n)+}g \times \Gamma^{(n)+}f$ is a disjoint union of simplicial maps $h_{p,q}: Z_{p,q} \to C_{p,q}$ given by

$$h_{p,q}[w, a_1, a_2, \ldots, a_p, b_1, \ldots, b_q] = [T(w), a_1, \ldots, a_p, b_1, \ldots, b_q]$$

where $a_i \in A$, $b_j \in B$, and where

$$T: W\Sigma_{p+q} \to W\Sigma_p \times W\Sigma_q$$

is the product $R_{\{1,2,\ldots,p\}} \times R_{\{p+1,\ldots,p+q\}}$ of reduction maps (2.4).

The restriction of T to a map $T: E_pC_{n+1}\Sigma_{p+q} \to C_{n+1}\Sigma_p \times C_n\Sigma_q$ is a $\Sigma_p \times \Sigma_q$-equivariant retraction. Hence $h_{p,q}: Z_{p,q} \to C_{p,q}$ is a retraction. To finish proving 4.11 we will show that $h_{p,q}$ is homotopic to the identity map $Z_{p,q} \to Z_{p,q}$.

A homotopy (4.14) $G: E_pC_{n+1}\Sigma_{p+q} \times I \to E_pC_{n+1}\Sigma_{p+q}$ is defined by

$$G(\alpha_0, \ldots, \alpha_k, 0) = (\alpha_0, \ldots, \alpha_k),$$

$$G(\alpha_0, \ldots, \alpha_k, 1) = (T(\alpha_0), \ldots, T(\alpha_k)) = T(\alpha_0, \ldots, \alpha_k),$$

$$G(\alpha_0, \ldots, \alpha_k, s_{k-1} \ldots \hat{s}_i \ldots s_0 e) = (T(\alpha_0), \ldots, T(\alpha_i), \alpha_{i+1}, \ldots, \alpha_k).$$

It is straightforward to check that G is a $\Sigma_p \times \Sigma_q$-equivariant homotopy such that G_0 (4.14) is the identity map and G_1 is the retraction

$$T: E_pC_{n+1}\Sigma_{p+q} \to C_{n+1}\Sigma_p \times C_n\Sigma_q,$$

Let $H: Z_{p,q} \times I \to Z_{p,q}$ be the homotopy defined by

$$H([w, a_1, \ldots, a_p, b_1, \ldots, b_q], t) = [G(w, t), a_1, \ldots, a_p, b_1, \ldots, b_q]$$

where $t \in I$ is a simplex. It follows from the previous paragraph that H is a well defined homotopy, that H_0 is the identity map $Z_{p,q} \to C_{p,q}$, and that H_1 is the retraction $h_{p,q}: Z_{p,q} \to C_{p,q}$. This completes the proof of 4.11 and also the proof of 4.7.

§5. Proof of Lemma 3.8(iv)

In this section we define the word length filtration, and use it and the filtration of $C_{n+1}\Sigma_p$ to prove 3.8(iv).

5.1. *The word length filtration of* Γ^+X [1, §6]. Let $m \geqq 0$ be an integer. Then $\Gamma_m^+ X$ is the simplicial subset of Γ^+X consisting of the simplices $[w, x_1, \ldots, x_p] \in \Gamma^+X$ such that $p \leqq m$. For integers $m \leqq n$, it follows that $\Gamma_m^+ X \subset \Gamma_n^+ X$, and that the union $\bigcup_{m \geq 0} \Gamma_m^+ X$ is Γ^+X.

5.2. *The induced filtrations*. The intersections $\Gamma_p^{(n)+}X = \Gamma^{(n)+}X \cap \Gamma_p^+ X$

give the word length filtration of $\Gamma^{(n)+}X$. Likewise $\Gamma^{(n)+}(X,A)$ is filtered by the intersections $\Gamma_p^{(n)+}(X,A)=\Gamma^{(n)+}(X,A)\cap\Gamma_p^+X$.

5.3. *The filtration quotients are denoted*:
$D_pX=\Gamma_p^+X/\Gamma_{p-1}^+X$,
$D_p^{(n)}X=\Gamma_p^{(n)+}X/\Gamma_{p-1}^{(n)+}X$,
$D_p^{(n)}(X,A)=\Gamma_p^{(n)+}(X,A)/\Gamma_{p-1}^{(n)+}(X,A)$.

Proof of 3.8(iv). We wish to prove that $T^{(n)}X=\Gamma^{(n)}(CX,X)$ is contractible. It follows from the results in [5, §5] that if the free simplicial monoid $\Gamma^{(n)+}(CX,X)$ is contractible, then its group completion $\Gamma^{(n)}(CX,X)$ is contractible. We show that $\Gamma^{(n)+}(CX,X)$ is contractible by proving that each filtration quotient $D_p^{(n)}(CX,X)$ is contractible.

Let k be an integer such that $0\leqq k\leqq p$. Let J_k denote the simplicial subset of $D_p^{(n)}(CX,X)$ consisting of the simplices $[w,x_1,\dots,x_p]\in D_p^{(n)}(CX,X)$ such that $w\in E_kC_{n+1}\Sigma_p$ (4.12). It follows from 4.13(i) that J_k is contained in J_{k+1}, that $J_0=D_p^{(n)}CX$, and that $J_p=D_p^{(n)}(CX,X)$. We show that $D_p^{(n)}(CX,X)$ is contractible by showing that each quotient J_k/J_{k-1}, $1\leqq b\leqq p$, is contractible.

5.4. We define a contracting homotopy $H:CX\times I\to CX$ (i.e. H_0 is the identity map and H_1 is the trivial map $CX\to\{*\}\subset CX$) by the formulas

$$H((p,x),0)=(p,x),\qquad (p,x)\in CX \text{ a simplex,}$$

$$H((p,x),1)=(k,*)\qquad (*\text{ basepoint of }X),$$

$$H((p,x),s_{k-1}\dots\hat{s}_i\dots s_0e)=(p,x),\qquad i+1\leqq p,$$

$$H((p,x),s_{k-1}\dots\hat{s}_i\dots s_0e)=(i+1,\partial_0^{i+1-p}x),\qquad p<i+1.$$

We use H to define a contracting homotopy of J_k/J_{k-1}. Each element $u\in J_k$ has by definition a representative of the form $[w,x_1,\dots,x_p]$ such that $w\in E_kC_{n+1}\Sigma_p$. It follows from a straightforward but tedious computation that u is in J_k but not in J_{k-1} if and only if the representative $u=[w,x_1,\dots,x_p]$ is unique up to the action of $\Sigma_p\times\Sigma_q$ (i.e. if $u=[v,y_1,\dots,y_p]$ and $v\in E_kC_{n+1}\Sigma_p$, then $v=w\cdot\alpha$ for some $\alpha\in\Sigma_p\times\Sigma_{p-k}$). A contracting homotopy $G_k:J_k/J_{k-1}\times I\to J_k/J_{k-1}$ is defined by

$$G_k(u,t)=[w,x_1,\dots,x_k,H(x_{k+1},t),\dots,H(x_p,t)]$$

where $t\in I$ is a simplex and $H:CX\times I\to CX$ is the contracting homotopy.

The homotopy G_k is well defined because of the uniqueness up to $\Sigma_k \times \Sigma_{p-k}$ action of the representative.

Appendix

In this appendix we consider the relationship between the two loop functors G (3.6) and Ω, and between the two suspension functors E and S.

The latter is easy.

PROPOSITION [7, p. 125]. *Let X be a pointed simplicial set. The realization $|EX|$ is naturally homeomorphic to $S|X|$.*

To relate the functors G and Ω we define a third loops functor L [7, p. 99]. Given a reduced simplicial set X satisfying the Kan extension condition [7, p. 2], a simplicial set LX having the homotopy type of loops on X is defined as follows. The k-simplices of LX are the $(k+1)$-simplices $x \in X_{k+1}$ such that $\partial_0 x = s_0^k *$ ($* \in X_0$ is the unique 0-simplex of X). A natural weak equivalence $f: LX \to GX$ is defined by $f(x) = \bar{x}$, the corresponding generator of GX.

The useful property of the functor L is that it commutes with the total singular functor sing [7, p. 2], in the sense that for a pointed space Y, there is a natural isomorphism

$$L \operatorname{sing} Y = \operatorname{sing} \Omega Y$$

(where we use a pointed version of sing).

The functors $|\ |$ and sing are adjoint and, for X a pointed simplicial set, there is a natural weak equivalence $X \to \operatorname{sing}|X|$ [7, §16].

Combining the last two paragraphs we obtain

PROPOSITION. *Let X be a reduced simplicial set. The following is a diagram of weak equivalences:*

$$GX \to G \operatorname{sing}|X| \leftarrow L \operatorname{sing}|X| = \operatorname{sing} \Omega |X|.$$

There are natural inclusions $\iota_X: X \to GEX$ (3.10) and $\iota_{|X|}: |X| \to \Omega S|X|$. The following proposition shows that these inclusions are naturally equivalent.

PROPOSITION. *Let x be a pointed simplicial set. The following diagram commutes and the indicated maps ($\xrightarrow{\sim}$) are weak equivalences:*

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "\wr"] & GEX \arrow[r, "\sim"] & G\,\mathrm{sing}|EX| & L\,\mathrm{sing}|EX| \arrow[l, "\sim"'] \arrow[d, equal] \\
\mathrm{sing}|X| \arrow[r] & \mathrm{sing}\,\Omega S|X| \arrow[rr, equal] & & \mathrm{sing}\,\Omega|EX|.
\end{tikzcd}
$$

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